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Doubly Linked Block Association Schemes

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ABSTRTACT

The inception of the Singly Linked Block (SLB) association scheme was known to due to Shrikhande [1], defining the association relationship of two blocks based on their intersection number of treatment which must be either 0 or 1. In the present work, starting from a given block design in which any two blocks intersect at either 1 or 2 treatment(s), a new 2-class association scheme to be named "Doubly Linked Block Association Scheme" is introduced and also some combinatorial approaches to its existence are laid down.

Keywords

Singly Linked Block (SLB) association scheme, λ^*-linked blocks, affine α-resolvable BIB design.

1. INTRODUCTION

Partially Balanced Incomplete Block (PBIB) designs were first introduced by Bose and Nair [2], in order to exhaust the strictly restricted condition that the concurrences of every two treatment should be same, as an extension of Balanced Incomplete Block (BIB) design. Nair and Rao [3] generalized the definition of PBIB designs, so as to include, as special cases, the cubic and other higher dimensional lattices. For precise and comprehensive analysis of the Partially Balanced Incomplete Block (PBIB) designs, later on it was realized that those designs should base on a abstract relationship configuration, so called Association Scheme. Further, introducing the concept of association schemes and basing the definition PBIB designs on these association schemes, the contributions to the existence of plan of Group Divisible, Triangular, Latin Square, Cyclic and Singly Linked Block (SLB) association schemes and that of plan of designs based on them with the specific range of parameters $r\leq10$ and $k\leq10$ due to Bose and Shimamoto [4] are available.

Among all the association schemes, it is hereby mentioned a few of them viz. Group Divisible, Triangular, Latin Square, Rectangular, Cyclic, Singly Linked Block (SLB) association schemes and others. SLB association scheme is defined on the existence of a BIB design with the pairwise balancing parameter $\lambda=1$. The existence of a SLB scheme is guarantee only when a BIB design with $\lambda=1$ exists. The inception of the SLB association was known to due to Shrikhande [1], defining the association relationship of two blocks based on their interaction number of treatment, which is either 0 or 1.

Youden [5] proposed that the dual of a BIB design having the parameters v^*, b^*, r^*, k^*, λ^* are so called "λ^*-linked blocks" on the reason that every pair of blocks of the dual design has λ^* treatments in common. When $\lambda^*=1(2)$, the dual design is called Singly(Doubly) Linked Blocks. For "Singly(Doubly) Linked Blocks", any two blocks of the given design have one(two) treatment(s) in common. It is known to all that the dual of any balanced incomplete block design in which any pair of treatments occurs λ^* times, guarantees that any two blocks of the dual have λ^* treatments in common. The dual of a BIBD are not necessarily always partially balanced. From the literature of Shrikhande [6] it is learnt that the dual of a BIBD (v^*, b^*, r^*, k^*, λ^*) will be partially balanced if $\lambda^*=1$ or $r^*=k$, $k^*=k-2$, $\lambda^*=2$ where k is the block-size of the dual design.

For the case $\lambda^*=1$, Shrikhande [6] and Roy [7] independently showed that such design is two associate classes with the first type parameters and the second type parameters,

$v= r^*(r^*k^*- r^*+1)/k^*$, $b=r^*k^*- r^*+1=v^*$, $r=k^*$, $k=r^*$, $\lambda_1=1$, $\lambda_2=0$; $n_1=k^*(r^*-1)$, $n_2= (r^*-k^*)(k^*-1)(r^*-1)/k^*$, $P^1_{11}=(r^*-2)+(k^*-1)^2$, $P^2_{11}= k^{*2}$.

This dual design under the case $\lambda^*=1$, is said to belong to the singly linked blocks (SLB). The association scheme on which the design bases, is given by the designs themselves, since any two treatments are first associate when and only when they occur in the same block. A certain overlap occurs between this type and the triangular type if $k^*=2$, $\lambda^*=1$. The design is the BIB design of Fisher and Yates [8] with parameters $v^*=n$, $b^*=n(n-1)/2$, $r^*=n-1$, $k^*=2$, $\lambda^*=1$. The dual of this BIB design is the SLB type design with parameters $v=n(n-1)/2$, $b=n$, $r=2$, $k=n-1$, $\lambda_1=1$,$\lambda_2=0$, $n_1=2n-4$, $P^1_{11}=n-1$, $P^2_{11}=4$.

For the case $\lambda^*=2$, the parameters of the dual i.e. PBIBD are $v=k^*(k^*-1)/2$, $b=(k^*-1)(k^*-2)/2$, $r=(k^*-2)$, $k=k^*$, $\lambda_1=1$,$\lambda_2=2$, $n_1=2k^*-4$, $P^1_{11}=k^*-2$, $P^2_{11}=4$. This design comes under the triangular design. We may call it Triangular doubly linked blocks as any two blocks interact at two common points, as shown in design, reference number T44, page 243, Clathworthy [9].

Let D* be a BIB design (v^*, b^*, r^*, k^*, $\lambda^*=1$). The dual of the given design is a Singly Linked Block design D, say with parameters $v=b^*$, $b=v^*$, $r=k^*$, $k=r^*$. Two of any new treatments are first associates if the corresponding blocks in D* have a common treatment and second associates if the corresponding blocks of D have no common treatment. Shrikhande [1] has shown that this relation satisfies the conditions of the association scheme for claiming it to be an association scheme. In general it is known as Singly Linked Block association scheme.

Given v treatments, viz., 1, 2, … , v, a relation satisfying the following conditions is said to be an association scheme with m classes:

(a) Any two treatments are either 1st, 2nd, … , or m-th associates, the relation of association being symmetrical, i.e., if the treatment α is an i-th associate of the treatment β, then β is also an i-th associate of the treatment α.

(b) Each treatment has n_i i-th associates, the number n_i being independent of α.

I If any two treatments are i-th associates, then the number of treatments which are the j-th associates of α and k-th associates of β is p_{jk}^{i} and is independent of the pair of i-th associates α and β.

The numbers v, n_i, p_{jk}^{i} (i, j, k=1, 2, …, m), are the parameters of the association scheme. They hold good the following relations, Bose and Nair [2]: $\sum_{i=1}^{m} n_i$=v-1, $\sum_{k=1}^{m} p_{jk}^{i}$=$n_j$-1 if i=j; = n_j otherwise, and $n_i p_{jk}^{i}$= $n_j p_{ik}^{j}$ = $n_k p_{ij}^{k}$.

2. DOUBLY LINKED BLOCK ASSOCIATION SCHEME

In the following it is being introduced the Doubly Linked Block association scheme, starting from a BIBD with pairwise balancing parameter value 2. Consider a BIB design D with parameters v*, b*, r*, k*, λ*=2 such that

(i) any two blocks intersect either one or two treatment(s) in common,

(ii) any block has n_i blocks which intersect i (=1,2) treatment(s) in common and no other blocks,

(iii) any two blocks having one(two) treatment(s) in common, have exactly t(s) blocks each of which intersects with each of the two blocks exactly one treatment in common.

Define two block numbers of D to be first associates if they have exactly one treatment in common and second associates if they have exactly two treatments in common. Then this association rule is an association scheme with two classes, which may be called Doubly Linked Block (DLB) association scheme. The existence of the association scheme can be insured only when the BIBD has the concurrence parameter value 2 and its blocks satisfy the above two conditions.

The parameters of the association scheme is given in the below given proposition.

2.1 Proposition

The parameters of the DLB association scheme are v=v^*, n_1=b^*-n_2-1=b^*-$\binom{k^*}{2}$-1, n_2=$\binom{k^*}{2}$, P_1=$\begin{bmatrix} t & n_1-t-1 \\ n_1-t-1 & n_2-n_1+t+1 \end{bmatrix}$, P_2=$\begin{bmatrix} s & n_1-s \\ n_1-s & n_2-n_1+s-1 \end{bmatrix}$.

Proof: As λ*=2, each of the $\binom{k^*}{2}$ pairs of the treatments arising from a block B of D, occurs once in some $\binom{k^*}{2}$ other blocks separately. Otherwise, there will be some blocks whose number of treatment intersection with the block B is 3,4 or more, violating the condition (i), 2 Section Further, any two blocks of D intersect either one or two treatment(s) in common. So, each of the b^*-$\binom{k^*}{2}$-1 remaining blocks intersect with B at one treatment in common by the condition (i), 2 Section. Then the proof can be completed from the assumption (ii) that any two blocks having one(two) treatment(s) in common, have exactly t(s) blocks each of which intersects with each of the two blocks exactly one treatment in common.

For an illustration of the 2.1 Proposition, let us take the Serial No. 4.4.1 of the BIB design, Dey [11].

2.1 Example: The BIB design has the parameters v=2t, b=4t-2, r=2t-1, k=t, λ=1.Taking t=3, the parameters of the BIB design become v^*=6, b^*=10, r^*=5, k^*=3, λ^*=1. The following is a plan of the BIB design with the parameters v^*=6, b^*=10, r^*=5, k^*=3, λ^*=2, where 1, 2, … , 6 represent the treatments and parentheses indicate the blocks. B_1=(1,2,3), B_2=(1,2,4), B_3=(1,3,5), B_4=(1,4,6), B_5=(1,5,6), B_6=(2,3,6), B_7=(2,4,5), B_8=(2,5,6), B_9=(3,4,6), B_{10}=(3,4,5). Then the parameters of the DLB association scheme are v=10, n_1=6, n_2=3, P_{11}^{1}=3, P_{11}^{2}=0.

2.1 Theorem

If there exists an affine α(>2) – resolvable BIB design with parameters v, b=βt, r=αt, k, λ=2, q_1=1, q_2 =2, then there exists a Doubly Linked Block (DLB) association scheme with parameters n_1=β-1, n_2=β(t-1), P_1=$\begin{bmatrix} \beta-2 & 0 \\ 0 & \beta(t-1) \end{bmatrix}$and P_2=$\begin{bmatrix} 0 & \beta-1 \\ \beta-1 & \beta(t-2) \end{bmatrix}$.

Proof: Consider an affine α-resolvable BIB design, D, with parameters v, b=βt, r=αt, k, λ=2, q_1=1, q_2 =2, (b>v). Then the b blocks can be divided into t sets, each of β blocks such that each treatment occurs α times in each set of blocks. Further any two blocks belonging to the same set contain q_1 (i.e. one) treatment in common and any two blocks belonging to the different sets contain q_2 (i.e. two) treatments in common.

Let S_i(i=1,2,…,t) be the i-th set containing β blocks viz., B_{i1}, B_{i2}, … ,$B_{i\beta}$, of D. Considering a block B_{im} belonged to S_i , there are β-1 blocks which intersects exactly one treatment in common with B_{im} and (t-1) sets other than S_i, each containing β blocks such that each of β(t-1) blocks in the other sets, apart from S_i, intersects exactly two treatments in common with B_{im}. Thus it is known that for a block B_{im} there are β-1 blocks and β(t-1) blocks which intersect only one treatment and two treatments in common with B_{im} respectively. So a block of the design intersects either one or two treatment(s) in common with any other blocks.

Consider two blocks B_{im} and $B_{im'}$ (m<m', m, m' = 1, 2 , … , β) belonged to the ith set, S_i, of D. They intersect exactly one treatment in common. Since q_1=1, each of the blocks B_{im} and $B_{im'}$ intersects only one treatment in common simultaneously with only those β-2 other blocks belonged to the same set S_i. No other block of the given affine α-resolvable BIB design D intersects one treatment in common with B_{im} and $B_{im'}$

simultaneously. That is, $\left|B_{im} \cap B_{im''}\right| = 1 = \left|B_{im'} \cap B_{im''}\right|$; $\forall$ m' =1, 2, ... , m-1, m+1, ... , m' -1, m' =+1, … , β. The number of blocks which intersect simultaneously one treatment in common with any two blocks with one treatment in common is β-2 . Thus P^1_{11} = β-2.

Further, consider any two blocks B_{il} and $B_{i'l'}$; i<i', l, l'=1, 2, ... , β ; which intersect exactly two treatments in common, i.e. they belong to different sets , S_i and $S_{i'}$ respectively of the affine α -resolvable BIB design D. Each of the two blocks B_{il} and $B_{i'l'}$ intersects exactly two treatments in common with those blocks which belonged to the other sets , viz , $S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_{i'-1}, S_{i'+1}, \ldots, S_t$. Then there does not exist any block $B_{i''l''} \in S_{i''}(i'' \neq i, i') \ni |B_{i''l''} \cap B_{il}| = |B_{i''l''} \cap B_{i'l'}| = 1$ in the design. In other words, the number of common blocks which intersect one treatment in common with two blocks (not particularly the same treatment) of two intersection treatments, is zero. Thus P^2_{11}= 0.

Thus the given affine α-resolvable BIB design satisfies the three conditions imposed on starting BIBD, for existence of Doubly Linked Block association scheme as given summarily:

(i) any two blocks intersect either one or two treatment(s) in common,

(ii) a block has n_i blocks which intersect i (=1, 2) treatment(s) in common and no other blocks,

(iii) any two blocks having one(two) treatment(s) in common, have exactly t(s) blocks each of which intersects with each of the two blocks exactly one treatment in common.

Defining two block numbers of the given affine α-resolvable BIB design with parameters v, b =βt, r=αt, k, λ=2, q_1=1, q_2=2, to be first associates if they have exactly one treatment in common and second associates if they have exactly two treatments in common, the parameters of Doubly Linked Block association scheme are n_1 = β-1, n_2 = β(t-1), $P_1 = \begin{bmatrix} \beta-2 & 0 \\ 0 & \beta(t-1) \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & \beta-1 \\ \beta-1 & \beta(t-2) \end{bmatrix}$. It completes the proof.

2.1 Remark

Since $q_2=k^2/v=2$, then $2v = k^2$. A necessary condition for the existence of the DLB association scheme on the affine α-resolvable BIBD(v, b=βt, r=αt, k, λ=2, q_1=1, q_2=2) is $v=2a^2$ for some a.

2.2 Remark

If α=1, the affine α-resolvable design has q_1=0. Thus it fails to contribute for existence of the DLB association scheme.

For any affine α-resolvable incomplete block design with parameters v, b =βt, r =αt , k , we have q_2=kα/β

i.e. 2β=kα since q_2 =2,

i.e. 2β=(β-1)α. …… (2.1)

As left hand side is even, the right hand side is so. Thus only two possibilities in the right side are there i.e. either (β-1) or α is even.

2.3 Remark

(i) If β is even, (β-1) is odd, so it compels α to be even. Further the left hand side of the relation (2.1) has co-factor of 4 i.e. 2^2 , the right hand side also has so. Thus α is of the form 4a for some a.

(ii) If β is odd, (β-1) is even. Thus α can never be even and also β is equal to 1+2a for some odd a, as the left hand side of the relation (2.1) has maximum exponential 2^1 i.e. 2.

2.4 Remark

A necessary condition of an affine α-resolvable BIBD(v, b=βt , r=αt , k, λ ,q_1=1 ,q_2 =2) to produce a DLB association scheme is that β-1 is divisible by α-1.

Then from the given affine α-resolvable BIBD, it is clear that 1= q_1=k(r-t)/(b-t). Therefore, k= (βt-t)/(αt-t), putting b=βt, r=αt

= (β-1)/(α-1) ;α>1 … (2.2)

If α=2,then k= β-1 and conversely. If either α=2 or k= β-1, the relation (2.1) gives that β=β-1, which is absurd. Thus we have a remark as follows.

2.5 Remark

A necessary condition of an affine α-resolvable BIBD(v, b= βt, r=αt, k, λ, q_1=1 ,q_2 =2) to produce a DLB association scheme is that k≠β-1 or α≠2. From the 2.2 Remark the necessary condition becomes more stronger as α>2.

For the future use, let us recall the 2.2 Theorem due to Shah [10] as a lemma at below.

2.1 Lemma (Shah[10])

A necessary and sufficient condition for an α-resolvable connected incomplete block design (v, b=βt, r=αt, k) to be affine α-resolvable is that (i) μ_0=k(b-r)/(b-t), (ii) k^2/v is a positive integer and (iii) k-μ_0 is a positive integer.

We consider here an affine α-resolvable connected incomplete block design with parameters v, b=βt , r=αt, k, q_1=1, q_2 =2. Let N be its incidence matrix and μ_0 be the largest characteristic root of NN' other than rk.

2.2 Theorem

A necessary condition of an affine α-resolvable connected incomplete block design (v, b=βt, r=αt, k, q_1=1, q_2 =2) to produce a Doubly Linked Block association scheme is that μ_0 is equal to (β-α)/(α-1).

Proof : From the condition (i) of the 2.1 Lemma, putting b=βt, r= αt, we have

μ_0 =(β- α)/(α-1) , since k=(β-α)/(α-1) by the relation (2.2),

which is an integer as both β-1 and α-1 are divisible by α-1.

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